October Math Gems

Problem of the week 1

§1 problems

Problem 1.1. In the following functional equation, solve for $f(x)$

$$f\left(x+\sqrt{x^2+1}\right)=\frac{x}{x+1}$$

---

answer. In this type of problems the easiest approach is to use substitution, so let:

$$x+\sqrt{x^2+1}=t \quad \textit{[1]}$$

This yields:

$$f(t)=\frac{x}{x+1}$$

Now, we need the function to be in terms of t, so we need to solve for x in *[1]*, by doing so we get:

$$x=\frac{t^2-1}{2t}$$

Then we'll substitute with this in the function:

$$f(t)=\frac{\frac{t^2-1}{2t}}{\frac{t^2-1}{2t}+1}$$

All the left is to do some simple algebraic manipulations.

$$f(t)=\frac{t^2-1}{t^2+2t-1}$$

Notice that there is no difference between x and t, they are just variables.

$$f(x)=\frac{x^2-1}{x^2+2x-1}$$

□

Problem 1.2. If

$$\frac{\log(a)}{b-c}=\frac{\log(b)}{c-a}=\frac{\log(c)}{a-b}$$

Then compute $a^a b^b c^c$.

---

answer. Let:
$$\frac{\log(a)}{b-c} = \frac{\log(b)}{c-a} = \frac{\log(c)}{a-b} = k$$
So we can get the following:
$$\log(a) = k(b-c)$$
$$\log(b) = k(c-a)$$
$$\log(c) = k(a-b)$$
Now:
$$\begin{aligned}\log(a^a b^b c^c) &= a\log(a) + b\log(b) + c\log(c) \\ &= a\left(k\left(b-c\right)\right) + b\left(k\left(c-a\right)\right) + c\left(k\left(a-b\right)\right) \\ &= 0\end{aligned}$$
This means that $a^a b^b c^c = 1$ □

Problem 1.3. let r be a real number such that
$$\sqrt[3]{r} + \frac{1}{\sqrt[3]{r}} = 3$$
Determine the value of
$$r^3 + \frac{1}{r^3}$$

---

answer. Note that by raising both sides to the power of 3, that we get:
$$r + \frac{1}{r} - 18 = 0$$
By doing the same in the last step, we get:
$$r^3 + \frac{1}{r^3} = 5778$$
□

Problem 1.4. Let a and b be distinct real numbers. Solve the following equation
$$\sqrt{x-b^2} - \sqrt{x-a^2} = a-b$$

---

answer. It should be obvious that the following conditions must hold true:
$$x \geq a^2 \quad x \geq b^2$$
Actually the simplest approach to solve this equation is taking the conjugate, another approaches leads to rather complicated computations. Taking the conjugate gives
$$\frac{a^2-b^2}{\sqrt{x-b^2}+\sqrt{x-a^2}} = a-b$$
which is equivalent to
$$\sqrt{x-b^2} + \sqrt{x-a^2} = a+b$$
Adding this to the original equation gives the following:
$$\sqrt{x-b^2} = a$$
This implies that
$$x = \sqrt{a^2+b^2}$$
□

Problem 1.5. Solve for all real solutions of x

$$(x^2 - 7x + 11)^{(x^2-13x+42)} = 1$$

answer. There are three cases where this equation can yield a solution of 1.

Case 1: The base is equal to 1

$$x^2 - 7x + 11 = 1$$

This equation give us the solutions 2 and 5.

Case 2: The exponent is equal to 0.

$$x^2 - 13x + 42 = 0$$

This equation give us the solutions 6 and 7. (Note: One must also make sure that these solutions don't make the base equal to 0 because this could give the unidentified case of 0^0)

Case 3: The base is equal to -1 and the exponent is an even number.

$$x^2 - 7x + 11 = -1$$

This yields the values 3 and 4 and indeed these values make the exponent even.

Therefore, the solutions for x are $2, 3, 4, 5, 6,$ and 7 □

Problem 1.6. A club has 30 members. The positions of president, vice president, and treasurer will be assigned to 3 distinct members. What is the maximum number of distinct assignments that can be made?

answer. We can use the law for permutations, which is $\frac{n!}{(n!-r!)}$ where n is the number of members and r is the number of people that you will choose. Therefore, the answer is 24360 □

Problem 1.7. If a, b, c are integers

$$\frac{ab}{a+b} = \frac{1}{3}, \qquad \frac{cb}{c+b} = \frac{1}{4}, \qquad \frac{ac}{a+c} = \frac{1}{5}$$

Find the value of

$$\frac{24abc}{ab+bc+ca}$$

answer.

$$\frac{a+b}{ab} = 3 \implies \frac{a}{b} + \frac{1}{a} = 3$$
$$\frac{b+c}{bc} = 4 \implies \frac{1}{c} + \frac{1}{b} = 4$$
$$\frac{ca}{c+a} = 5 \implies \frac{1}{c} + \frac{1}{a} = 5$$

Sum these three equations

$$2(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}) = 12 \implies (\frac{1}{a} + \frac{1}{b} + \frac{1}{c}) = 6$$
$$\frac{abc}{bc+ac+ab} = \frac{1}{6}$$
$$\frac{24abc}{bc+ac+ab} = \frac{24}{6} = 4$$

□

Problem 1.8. The general solution of

$$\sin x - 3\sin^2 x + \sin^3 x = \cos x - 3\cos^2 x + \cos^3 x$$

---

answer.

$$(\sin x + \sin 3x) - 3\sin 2x = (\cos x + \cos 3x) - 3\cos 2x$$

$$2\sin 2x \cos x - 3\sin 2x = 2\cos 2x \times \cos x - 3\cos 2x$$

$$\sin 2x(2\cos x - 3) = \cos 2x(2\cos x - 3)$$

$$\sin 2x = \cos 2x \qquad \cos x \neq \frac{3}{2}$$

$$\tan 2x = 1 \implies x = \frac{n\pi}{2} + \frac{\pi}{8} \qquad (\text{As } \frac{\sin 2x}{\cos 2x} = \tan 2x)$$

□

Problem 1.9. If

$$f(n+3) = \frac{f(n) - 1}{f(n) + 1}, \qquad f(11) = 11$$

Find the value of $f(2003) =$

---

answer. Suppose that $n = 11$ so $n + 3 = 14$

$$f(11+3) = f(14) = \frac{f(11) - 1}{f(11) + 1} = \frac{11 - 1}{11 + 1} = \frac{5}{6}$$

Now, we can find $f(17)$ by using the value of $f(14)$

$$f(14+3) = f(17) = \frac{f(14) - 1}{f(14) + 1} = \frac{\frac{5}{6} - 1}{\frac{5}{6} + 1} = \frac{-1}{11}$$

$$f(17+3) = f(20) = \frac{f(17) - 1}{f(17) + 1} = \frac{\frac{-1}{11} - 1}{\frac{-1}{11} + 1} = \frac{-6}{5}$$

$$f(20+3) = f(20+3) = \frac{f(20) - 1}{f(20) + 1} = \frac{\frac{-6}{5} - 1}{\frac{-6}{5} + 1} = 11$$

you must have noticed the pattern, we return to $f(x) = 11$ after 12 rounds, we can suppose that

$$2003 = 11 + 12k$$

and if we get the value of k as an integer number so $f(2003) = 11$. Now, solve for k,

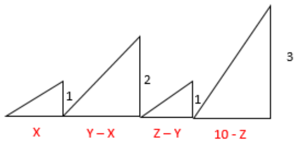
$$2003 = 11 + 12k \implies k = 166$$

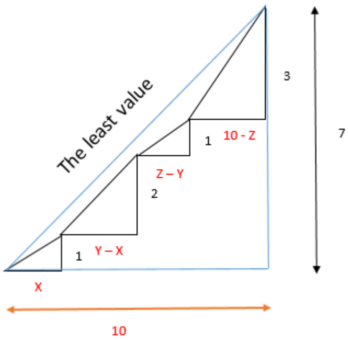
So, we can say that $f(2003) = 11$ □

Problem 1.10. Find the least value of this algebraic expression

$$\sqrt{x^2 + 1} + \sqrt{(y - x)^2 + 4} + \sqrt{(z - y)^2 + 1} + \sqrt{(10 - z)^2 + 9}$$

$$\sqrt{x^2+1}+\sqrt{(y-x)^2+4}+\sqrt{(z-y)^2+1}+\sqrt{(10-z)^2+9}$$





answer. We can say the least value of this algebraic expression is the hypotenuse of the right-angled-triangle

$$\sqrt{(7)^2+(10)^2}=\sqrt{149}$$

□

Problem 1.11. If $a+b+c=0$ then the value of

$$\frac{a^7+b^7+c^7}{abc(a^4+b^4+c^4)}$$

---

answer.

$$a+b+c=0 \implies (a+b+c)^2=0$$

$$(a+b+c)^2=a^2+b^2+c^2+2(ab+ac+bc)=0$$

$$(a^2+b^2+c^2)^2=(-2(ab+ac+bc))^2 \to *$$

$$(a^2+b^2+c^2)^2=a^4+b^4+c^4+2(a^2b^2+a^2c^2+b^2c^2) \to (1)$$

$$(-2(ab+ac+bc))^2=4[(a^2b^2+a^2c^2+b^2c^2)+2(ab^2c+a^2bc+bc^2a)]=4[(a^2b^2+a^2c^2+b^2c^2)] \to (2)$$

substitute (1) and (2) in (*), we get

$$a^4+b^4+c^4=2(a^2b^2+a^2c^2+b^2c^2) \to (3)$$

$$\because a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ac)=0$$

$$\therefore a^3+b^3+c^3=3abc \to (4)$$

Now, multiply (3) and (4)

$$(a^4+b^4+c^4)(a^3+b^3+c^3)=a^7+b^c+c^7-abc(a^2b^2+b^2c^2+a^2c^2)=6abc(a^2b^2+b^2c^2+a^2c^2)$$

$$\therefore a^7+b^c+c^7=7abc(a^2b^2+b^2c^2+a^2c^2) \to (5)$$

substitute with (3) and (5) in the needed expression

$$\frac{(a^7+b^7+c^7)}{abc(a^4+b^4+c^4)}=\frac{7abc(a^2b^2+b^2c^2+a^2c^2)}{7abc(a^2b^2+b^2c^2+a^2c^2)}=\frac{7}{2}$$

□

Problem 1.12.

$$\frac{1}{\log_2 m}+\frac{1}{\log_4 m}+\frac{1}{\log_8 m}......+\frac{1}{\log_{2^n} m}=n(n+1)\alpha$$

Find α

---

answer.

$$n(n+1)\alpha=\frac{1}{\log_2 m}+\frac{1}{\log_4 m}+\frac{1}{\log_8 m}....+\frac{1}{\log_{2^n} m}$$

$$\log_m 2+2\log_m 2+3\log_m 2....+n\log_m 2$$

$$(1+2+3....+n)\log_m 2=\frac{n(n+1)}{2}\log_m 2=n(n+1)\frac{1}{2}\log_m 2$$

$$n(n+1)\log_m \sqrt{2}.$$

Then $\alpha=\log_m \sqrt{2}$ □

Problem 1.13. In the triangle ABC, $AC=2BC$, $\angle C=90$ and D is the foot of the altitude from C onto AB. A circle diameter AD intersects the segment AC at E. Find $AE:EC$.

---

answer.

$$\triangle ADC\sim\triangle CDB \implies \frac{AD}{CD}=\frac{CD}{DB}=2$$

So,

$$CD=2DB,\quad CD=\frac{1}{2}AD \implies AD=4DB$$

$\angle AED$ is 90, as AD is a diameter. So,

$$DE\parallel BC \implies \triangle AED\sim\triangle ACB$$

$$\frac{AE}{AC}=\frac{ED}{CB}=\frac{AD}{AB} \implies \frac{AE}{AC}=\frac{4}{5} \implies \frac{AE}{EC}=\frac{4}{1}$$

□

Problem 1.14. Evaluate the sum of the expression

$$\sum_{i=1}^{n} i(2)^i$$

---

answer.

$$\sum_{i=1}^{n} i(2)^i=2+2*2^2+3*2^3....n*2^n$$

$$\sum_{i=1}^{n}(2)^i+\sum_{i=2}^{n}(2)^i+\sum_{i=3}^{n}(2)^i.....+2^n$$

$$2^{n+1}-2+2^{n+1}-2^2+2^{n+1}-2^3+....+2^{n+1}-2^n+n2^{n+1}-\sum_{i=1}^{n}(2)^i$$

$$=n2^{n+1}-(2^{n+1}-2)=(n-1)2^{n+1}+2$$

□

Problem 1.15. If
$$\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$$
Then, find the value of
$$x^{100} + y^{100} + z^{100} - \frac{9}{x^{101} + y^{101} + z^{101}}$$

---

answer. Notice that the range of $\sin^{-1}$ is $[\frac{-\pi}{2}, \frac{\pi}{2}]$ Therefore
$$\sin^{-1} x, \sin^{-1} y, \sin^{-1} z \leq \frac{\pi}{2}$$
$$\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$$
Hence,
$$\sin^{-1} x = \sin^{-1} y = \sin^{-1} z = \frac{\pi}{2}$$
$$x^{100} + y^{100} + z^{100} - \frac{9}{x^{101} + y^{101} + z^{101}} = (1+1+1) - \frac{9}{1+1+1} = 0$$
□

Problem 1.16. If four points are $A(6,3), B(-3,5), C(4,-2), P(x,y)$, then the ratio of areas between triangle ABC, PBC is

---

answer. Note that area of triangle
$$PBC = \frac{1}{2}(x(y_2 - y_3) + x_3(y - y_2) + x_2(y_3 - y)) =$$
$$\frac{1}{2}(x(5+2) + 4(y-5) - 3(-2-y)) = \frac{1}{2}(7x + 7y - 14)$$
Similarly, the area of
$$ABC = \frac{1}{2}(6(5+2) - 3(-2-3) + 4(3-5)) = \frac{49}{2}$$
Hence, the ratio=
$$\frac{7x + 7x - 14}{49} = \frac{x + y - 2}{7}$$
□

Problem 1.17. Geeta and Babeeta play the following game. An inclusive integer between 0 and 999 is selected and given to Geeta. Whenever Geeta receives a number, he doubles it and passes the result to Babeeta. Whenever Babeeta receives a number, she adds 50 to it and passes the result to Geeta. The winner is the last person who produces a number less than 1000. Let N be the smallest initial number that results in a win for Geeta. What is the sum of the digits of N?

---

answer. The pattern will be

$$2N, 2N+50, 4N+100, 4N+150, 8N+300, 16N+700, 16N+750$$

So, we have

$$16N+750 > 1000 > 16N+700 \implies 19 > N > 15$$

as in the question wanted the smallest positive integer so the answer is 16. □

Problem 1.18. A function f is defined recursively by $f(1) = f(2) = 1$ and $f(n) = f(n-1) - f(n-2) + n$ for all integers $n \geq 3$ what is the sum of the digits of $f(2018)$?

---

answer.

$$f(3) = f(2) - f(1) + 3 = 3$$
$$f(4) = f(3) - f(2) + 4 = 6$$
$$f(5) = f(4) - f(3) + 5 = 8$$
$$f(6) = f(5) - f(4) + 6 = 8$$
$$f(7) = f(6) - f(5) + 7 = 7$$
$$f(8) = f(7) - f(6) + 8 = 7$$
$$f(9) = f(8) - f(7) + 9 = 9$$

We can notice that $f(n) = n$ when n is an odd number and dividable by 3. the pattern of the number is $2, 3, 2, 0, -1, 0,$

$$f(2013) = 2013$$

as is it odd number and divisible by three, the pattern will be $+3, +2, +0, -1, +0,$

$$f(2014) = 2013 + 3 = 2016$$
$$f(2015) = 2016 + 2 = 2018$$
$$f(2016) = 2018 + 0 = 2018$$
$$f(2017) = 2018 - 1 = 2017$$
$$f(2018) = 2017 + 0 = 2017$$

□

Problem 1.19. Known that $a, b, c, d > 0$

$$\frac{1}{a+1} + \frac{2}{b+1} + \frac{3}{c+1} + \frac{4}{d+1} = x$$
$$\frac{a}{a+1} + \frac{2b}{b+1} + \frac{3c}{c+1} + \frac{4d}{d+1} = 4x$$

Then, $4x^2$ is equal to

---

answer. Add both equations

$$\frac{a+1}{a+1} + \frac{2(b+1)}{b+1} + \frac{3(c+1)}{c+1} + \frac{4(d+1)}{d+1} = 5x$$
$$1+2+3+4 = 5x \qquad \rightarrow \qquad x = 2$$

Which yeilds the answer to 16 □

Problem 1.20. If x is a complex number satisfying $x^2 + x + 1 = 0$, find the value of

$$x^{49} + x^{50} + x^{51} + x^{52} + x^{53}$$

answer. First, we note that $x^2 + x + 1 = \frac{x^3-1}{x-1}$. Therefore, we see that $x^3 = 1$. Now, we can rewrite the given expression as

$$x^{49}(1 + x + x^2) + x^{51}(x + x^2)$$
$$= 0 + (x^3)^{17}(-1)$$
$$= -1$$

□